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A Geometrical Characterization of the $C(K)$ and $C_0(K)$ Spaces

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In this paper we present, among other related results, the following geometrical characterization of $C(K)$ and $C_0(K)$ spaces. Let X be a Banach space. Let $r(A)$, $r_G(A)$, and $E^0(A)$ denote the Chebyshev radius of A, the Chebyshev radius of A relative to G, and the set of Chebyshev centers of A, respectively. We prove that X is isometric to a $C(K)$ or $C_0(K)$ space if and only if $r_G(A) = r(A) + \text{dist}(E^0(A), G)$ for every nonempty bounded subset A and nonempty subset G of X.

Key Words: Banach lattice; Chebyshev element; hyperconvex metric space; spaces of continuous functions.

1. INTRODUCTION

We use the following notation and definitions. Let M be a metric space, A a nonempty bounded subset of M, and G a nonempty subset of M. Let $B(x, r)$ denote the closed ball of center $x \in M$ and radius r. The Chebyshev radius of A, $r(A)$ is the infimum of all numbers $r > 0$ for which there exists $y \in M$ such that $A \subset B(y, r)$. Similarly, the relative Chebyshev radius of A with respect to G, $r_G(A)$, is the infimum of all numbers $r > 0$ for which there exists $y \in G$ such that $A \subset B(y, r)$. The possibly empty set of all points

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$y \in M$ for which $A \subseteq B(y, r(A))$ is called the set of Chebyshev centers of A and is denoted by $E^0(A)$. The nonempty set of all points $y \in M$ for which $A \subseteq B(y, r(A)+\varepsilon)$ with $\varepsilon > 0$ is called the set of ε-Chebyshev centers of A and is denoted by $E^\varepsilon(A)$. Finally, $\operatorname{dist}(A, G)$ stands for the infimum of all numbers $d(x, y)$ where $x \in A$ and $y \in G$.

The notation $C_0(K)$ will be used in its standard way, i.e., it will denote the space of continuous real functions, which vanish at infinity, defined on a locally compact Hausdorff space K with the usual supremum norm. When K is considered to be a compact Hausdorff space then we write $C(K)$.

In 1975, Smith and Ward [7, Theorem 2.2] proved that $r_G(A) = r(A) + \operatorname{dist}(E^0(A), G)$ whenever A and G are nonempty subsets of a $C(K)$ space with A bounded. The main result of our paper, stated below, completes this in the sense that it characterizes all Banach spaces for which the above equation holds.

THEOREM. *Let X be a Banach space. Then the following assertions are equivalent*:

1. $r_G(A) = r(A) = \operatorname{dist}(E^0(A), G)$ *for every nonempty bounded subset A and nonempty subset G of X.*
2. $r_G(A) = r(A) + \lim_{\varepsilon \to 0} \operatorname{dist}(E^\varepsilon(A), G)$ *for every nonempty bounded subset A and nonempty subset G of X.*
3. *X is isometric to either a $C(K)$ space or a $C_0(K)$ space.*

The main work is divided into three sections. In Section 2, we recall some essential results from Banach lattice theory and the Lifschitz modulus introduced in [8] by two of the present authors, as well as some other results related to the properties of intersections of balls which were studied in [5, 6]. In Section 3, we derive the main characterization result of this work. Finally, in Section 4, we extend the results of Section 3 to incomplete normed linear spaces and we also obtain some new properties of the Lifschitz modulus.

2. PREVIOUS RESULTS

In this section we recall some definitions and results that will be needed throughout our exposition. We refer to [4] for standard definitions and notations on Banach lattice theory. The next two theorems are due to Kakutani (see [4] for proofs) and will be used as a final step in our characterization result.

THEOREM 2.1. *If a closed sublattice Y of a space of continuous functions $C(K)$ separates the points of K (i.e., for every different t_1 and t_2 in K there exists f in Y such that $f(t_1) \neq f(t_2)$) and contains the constant function* 1, *then $Y = C(K)$.*

THEOREM 2.2. *Let Y be a closed linear subspace of a space of continuous functions $C(K)$. Let $\mathfrak{T}_Y$ denote the collection of al triplets (t_1, t_2, α), with t_1, $t_2 \in K$ and α a real positive number, such that $f(t_1) = \alpha f(t_2)$ for all $f \in Y$. Then Y is a sublattice of $C(K)$ if and only if Y contains the set of all functions f in $C(K)$ such that $f(t_1) = \alpha f(t_2)$ for every triplet (t_1, t_2, α) in $\mathfrak{T}_Y$.*

The following corollary follows in a direct way from the previous theorem.

COROLLARY 2.3. *Let Y be a sublattice of a $C(K)$ such that it separates the points of K. Then Y is isometric to $C_0(\Omega)$, for a certain locally compact Hausdorff space Ω, if and only if*

$$\mathfrak{T}_Y = \{(t, t, 1) : t \in K\} \cup \{(t_0, t_0, \alpha) : t_0 \in K, \alpha \in \mathbb{R}^+\}$$

for some $t_0 \in K$.

In this work we will also deal with properties of intersections of balls. The following definitions were introduced in [1].

DEFINITION 2.4. A metric space M is said to be hyperconvex if given any family $\{x_\alpha : \alpha \in \mathscr{A}\}$ of points of M and any family $\{r_\alpha : \alpha \in \mathscr{A}\}$ of real positive numbers satisfying $d(x_\alpha, x_\beta) \leqslant r_\alpha + r_\beta$ for every α and $\beta \in \mathscr{A}$, it is the case that $\bigcap_{\alpha \in \mathscr{A}} B(x_\alpha, r_\alpha) \neq \varnothing$.

In addition, if $\aleph$ is a cardinal number, we will say that a metric space M is an $\aleph$-*hyperconvex* space if the above definition is verified in M whenever the cardinality of the set of centers $\{x_\alpha : \alpha \in \mathscr{A}\}$ is strictly less than $\aleph$. In case $\aleph$ is the cardinal number of the natural numbers we will say that M is an $\aleph_0$-*hyperconvex* space. M will be said to be an R-$\aleph$-*hyperconvex* space if for every subset A of M with cardinality strictly less than $\aleph$ and for every $r > \frac{1}{2}\operatorname{diam}(A)$, the intersection $\bigcap \{B(x, r) : x \in A\}$ is nonempty. Finally, to avoid confusion, it should be noted that $(n+1)$-hyperconvexity is equivalent to the n.2.I.P. considered in [5] and [6].

The following two results can be found in [6], the second also in [5].

THEOREM 2.5. *Let X be a Banach space. Then X is $\aleph_0$-hyperconvex if and only if X is isometric to a closed sublattice Y of a space of continuous functions $C(K)$ which separates the points of K.*

THEOREM 2.6. *A Banach space is an R-$\aleph_0$-hyperconvex space if and only if it is an $\aleph_0$-hyperconvex space.*

DEFINITION 2.7 [8]. Let M be a metric space; then, for every $d \geqslant 0$, the Lifschitz modulus of M on d, $\tilde{\kappa}_M(d)$, is the supremum of all the positive real numbers k such that there exists $\alpha \in (0, 1)$ so that for every pair of points x and y in M, and for every $r \in \mathbb{R}^+$ there exists a point z in M such that $d(z, y) \leqslant \alpha\, dr$ and $B(x, r) \cap B(y, kr) \subseteq B(z, r)$.

LEMMA 2.8. *If M is a nonsingleton metric space, then*

$$\max\{1, d-1\} \leqslant \tilde{\kappa}_M(d) \leqslant d+1$$

for all $d \geqslant 0$.

The following result is a minor modification of Theorem 4.2 in [8].

THEOREM 2.9. *Let A and G be two nonempty subsets of a normed space X with A nonsingleton and bounded. Then*

$$r_G(A) \geqslant r(A)\, \tilde{\kappa}_X\left(\frac{d_r(A, G)}{r(A)}\right),$$

where

$$d_r(A, G) = \begin{cases} \operatorname{dist}(E^0(A), G), & \text{if } E^0(A) \neq \varnothing \\ \lim_{\varepsilon \to 0^+} \operatorname{dist}(E^\varepsilon(A), G), & \text{if } E^0(A) = \varnothing. \end{cases}$$

The following lemma is a direct consequence of the triangle inequality.

LEMMA 2.10. *Let A and G be two nonempty subsets of a metric space M with A bounded, then $r_G(A) \leqslant r(A) + \lim_{\varepsilon \to 0^+} \operatorname{dist}(E^\varepsilon(A), G)$.*

COROLLARY 2.11. *Let X be a normed space such that $\tilde{\kappa}_X(d) = d+1$ for all positive d. If A and G are two nonempty subsets of X with A bounded, then*

$$r_G(A) = r(A) + \lim_{\varepsilon \to 0^+} \operatorname{dist}(E^\varepsilon(A), G).$$

If, in addition, $E^0(A)$ is nonempty, then $r_G(A) = r(A) + \operatorname{dist}(E^0(A), G)$.

The next definition will play a main role in all our exposition.

DEFINITION 2.12. We shall say that a metric space M is a Smith–Ward space if

$$r_G(A) = r(A) + \lim_{\varepsilon \to 0^+} \operatorname{dist}(E^{\varepsilon}(A), G) \tag{2.1}$$

for each nonempty bounded subset A and each nonempty subset G of M.

Note that whenever $E^0(A) \neq \emptyset$ in the above definition the fact that $r_G(A) = r(A) + \operatorname{dist}(E^0(A), G)$ implies (2.1).

3. MAIN RESULTS

Our first result of this section states that hyperconvex metric spaces are Smith–Ward spaces.

THEOREM 3.1. *If M is a hyperconvex metric space then $\tilde{\kappa}_M(d) = d + 1$ for all positive numbers d, and*

$$r_G(A) = r(A) + \operatorname{dist}(E^0(A), G)$$

for each nonempty bounded subset A and each nonempty subset G of M.

Proof. In order to prove that $\tilde{\kappa}_M(d) = d + 1$ it will be enough to prove that $\tilde{\kappa}_M(d) \geqslant k$ for every $k < d + 1$. So, given $k < d + 1$, we fix $\alpha < 1$ so that $\alpha d + 1 \geqslant k$. Then for every $v \in B(x, r) \cap B(y, kr)$ we have $d(v, y) \leqslant \alpha\, dr + r$ and hence, by the hyperconvexity of M,

$$B(y, \alpha\, dr) \cap \left(\bigcap \{ B(v, r) : v \in B(x, r) \cap B(y, kr) \} \right) \neq \emptyset.$$

We get $\tilde{\kappa}_M(d) \geqslant k$ by taking z as any element in this intersection.

From Corollary 2.11 we may conclude that hyperconvex Banach spaces are Smith–Ward spaces. In order to extend this result to the metric context we consider M isometrically embedded into $\ell^{\infty}(I)$ for a certain set of indexes I. Then for every nonempty bounded subset A of M

$$E^{\varepsilon}_{\infty}(A) \supseteq E^{\varepsilon}_{m}(A), \tag{3.1}$$

where the first Chebyshev center is taken with respect to $\ell^{\infty}(I)$ and the second one with respect to M (note that ℓ^{∞} spaces are hyperconvex and that the Chebyshev radius of a subset of a hyperconvex space is the same whatever the hyperconvex space is). It is straightforward to prove that $\operatorname{dist}(E^{\varepsilon}_{\infty}(A), G) = \operatorname{dist}(E^{\varepsilon}_{M}(A), G)$. Finally, by the hyperconvexity of M,

$$E^{0}(A) = \bigcap \{B(x, r') \colon x \in A, r' > r(A)\} \neq \emptyset,$$

so $r_G(A) = r(A) + \operatorname{dist}(E^0(A), G)$, which completes the proof. ∎

Remark 3.2. Note that the main result of this paper (Theorem 3.11) shows that the converse of Theorem 3.1 does not hold. Since our main goal is to characterize Banach spaces which are Smith–Ward spaces, very little is said about the equivalent problem in the setting of metric spaces. Nevertheless some information about the metric structure of such spaces is provided by Corollary 3.5.

One of the main results of this section states that every Smith–Ward Banach space is an $\aleph_0$-hyperconvex space. Furthermore the next example is interesting. We recall a well-known example of a 4-hyperconvex space which is not 5-hyperconvex (note that it was proved in [6] that every 5-hyperconvex Banach space is $\aleph_0$-hyperconvex). Consequently, it is a little surprising that this space is not only a non-Smith–Ward space but also the value of $\tilde{\kappa}$ in this space is the lowest possible (see Lemma 2.8) for all $d \geqslant 3$.

Example 3.3. Let $X = (\mathbb{R}^3, \|\cdot\|_1)$. It is shown in [6] that this space is 4-hyperconvex but not $\aleph_0$-hyperconvex. Consider the sets $A = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and $G = \{(1, 1, 1)\}$. It is not hard to prove that $r(A) = 1$ and $E^0(A) = \{(0, 0, 0)\}$. Then

$$4 = r(A) + \lim_{\varepsilon \to 0^+} \operatorname{dist}(E^{\varepsilon}(A), G) > r_G(A)$$

$$= \lim_{\varepsilon \to 0^+} \operatorname{dist}(E^{\varepsilon}(A), G) - r(A) = 3 - 1 = 2.$$

Therefore this space is not a Smith–Ward space. Moreover, $\tilde{\kappa}_X(d) = d - 1$ for all $d \geqslant 3$. Indeed, for $d \geqslant 3$ take $x = (0, 0, 0)$ and $y = (\frac{d}{3}, \frac{d}{3}, \frac{d}{3})$. Then $A \subseteq B(x, 1) \cap B(y, d-1)$, and if z is such that $B(x, 1) \cap B(y, 2) \subseteq B(z, 1)$ then $z = x$.

Theorem 3.4. *If X is a Smith–Ward Banach space then it is an $\aleph_0$-hyperconvex space.*

Proof. By Theorem 2.6 it will be enough to prove that X is an R-$\aleph_0$-hyperconvex space. Let $\{x_1, x_2, ..., x_n\}$ be a finite collection of points in X. Let $R = \max\{\|x_i - x_j\| : 1 \leqslant i, j \leqslant n\} = \|x_1 - x_2\|$. We fix $r = \frac{R}{2}$. Then r is less than or qual to the Chebyshev radius of $\{x_1, x_2, ..., x_i\}$ for all

$i \in \{3, ..., n\}$. As the first step of the proof we take $A = \{x_1, x_2\}$ and $G = \{x_3\}$. Then, from the fact that X is a Smith–Ward space,

$$r_{\{x_3\}}(\{x_1, x_2\}) = r(\{x_1, x_2\}) + \lim_{\varepsilon \to 0^+} \text{dist}(E^{\varepsilon}(\{x_1, x_2\}), \{x_3\}).$$

Hence $2r \geqslant r + \lim_{\varepsilon \to 0^+} \text{dist}(E^{\varepsilon}(\{x_1, x_2\}), \{x_3\})$, and

$$B(x_1, r+\varepsilon) \cap B(x_2, r+\varepsilon) \cap B(x_3, r+\varepsilon) \neq \varnothing$$

for every $\varepsilon > 0$. So, in addition, $r(\{x_1, x_2, x_3\}) = r$.

Let A now be $\{x_1, x_2, x_3\}$ and $G = \{x_4\}$. In a similar way to before we obtain that

$$B(x_1, r+\varepsilon) \cap B(x_2, r+\varepsilon) \cap B(x_3, r+\varepsilon) \cap B(x_4, r+\varepsilon) \neq \varnothing$$

for every $\varepsilon > 0$ and so $r(\{x_1, x_2, x_3, x_4\}) = r$.

We proceed in that way until $A = \{x_1, x_2, ..., x_{n-1}\}$ and $G = \{x_n\}$. Then, by the arbitrariness of ε, the desired result follows. ∎

COROLLARY 3.5. *If M is a Smith–Ward space then it is an R-$\aleph_0$-hyperconvex space.*

Our next goal is to study which $\aleph_0$-hyperconvex Banach spaces are Smith–Ward spaces.

DEFINITION 3.6. Let Y be a sublattice of a $C(K)$ space; then we call $\Omega_Y = \{t \in K :$ if $y \in Y$ and $y(t) = \|y\|$ then $y = 0$, and there exists $y_0 \in Y$ such that $y_0(t) > 0\}$.

DEFINITION 3.7. Given a Banach space X isometric to a sublattice of a $C(K)$ space, we will say that Y is a proper representation of X in $C(K)$ if it is a sublattice of $C(K)$ isometric to X, the topological interior of Ω_Y is empty, separates the points of K and if t_0 is an element of K for which every $y \in Y$ vanishes, then the singleton $\{t_0\}$ is not an open set in K.

The existence of these proper representations is given by the following lemma.

LEMMA 3.8. *If X is a Banach space isometric to a sublattice of a $C(K)$ space that separates the points of K, then there exists a compact Hausdorff space K' such that we can find a proper representation of X in $C(K')$.*

Proof. Let Y be the sublattice of $C(K)$ isometric to X given by the statement. Let us denote by $\mathrm{int}(\Omega_Y)$ the topological interior of Ω_Y in K. Since K is compact, then the set

$$K' = K \setminus (\mathrm{int}(\Omega_Y) \cup \mathrm{int}\{t \in K : y(t) = 0 \text{ for all } y \in Y\})$$

is also a compact set. If we consider the action of restricting each element of Y to K' then we obtain an isometry from Y into its image. The image of Y is the desired proper representation of X in $C(K')$. ∎

The following corollary is immediate from Theorem 2.5.

COROLLARY 3.9. *Every $\aleph_0$-hyperconvex Banach space has a proper representation in a certain $C(K)$ space.*

The next theorem gives the first characterization for a Banach space to be a Smith–Ward space.

THEOREM 3.10. *Let X be a Banach space. Then the following assertions are equivalent*:

(a) $\tilde{\kappa}_X(d) = d + 1$ *for all positive numbers d.*

(b) *X is a Smith–Ward space.*

(c) *X is an $\aleph_0$-hyperconvex space such that if Y is a proper representation of X then $x \wedge k$ is in Y for each positive function x in Y and for each positive real number k such that*

$$\max\{\|x\| - 1, \min_{t \in K} x(t)\} < k < \|x\|.$$

(d) *There exists a $C(K)$ space where X has a proper representation, and if Y is a proper representation of X in $C(K)$ then Ω_Y is the empty set.*

(e) *There exists a $C(K)$ space where X has a proper representation, and if Y is a proper representation of X in $C(K)$ then $x \wedge k$ belongs to Y for each positive element x in Y and each positive real number k.*

Proof. (a) ⇒ (b) This implication was already stated in Corollary 2.11.

(b) ⇒ (c) Since X is a Smith–Ward Banach space, it is an $\aleph_0$-hyperconvex space. So we may fix Y as a proper representation of K in a certain $C(K)$ space. Let us denote Ω_Y simply by Ω.

Let x be a positive element of Y. It can be assumed, without loss of generality, that x is not a constant function (otherwise (c) follows directly from the linear properties of Y). Let $K_0 = \{t \in K : y(t) \neq 0 \text{ for some } y \in Y\}$.

We fix k as in the statement of (c). Then we consider $A = B(x, 1) \cap B(0, k+1) \cap Y$ and $G = \{0\}$, where 0 stands for the null function. We first prove that $r(A) = 1$ and $r_G(A) = k+1$. We claim that

$$(K_0 \backslash \Omega) \cap \{s \in K : x(x) \leqslant k\}$$

is nonempty and that for every t in it there exist p_t and q_t in A such that $p_t(t) = x(t) + 1$ and $q_t(t) = x(t) - 1$. In fact, since Y is a proper representation of X the set $\{t \in K : y(t) = 0 \text{ for all } y \in Y\}$ must be empty or singleton. If it is a singleton, denoted by $\{t_0\}$, then it cannot be an open set. So $\Omega \cup \{t_0\}$ has an empty interior. Now, since $k > \min_{s \in K} x(s)$, we may conclude that $(K_0 \backslash \Omega) \cap \{s \in K : x(s) \leqslant k\}$ is nonempty.

Consider a fixed t in $(K_0 \backslash \Omega) \cap \{s \in K : x(s) \leqslant k\}$. Since t is in $K_0 \backslash \Omega$, there is a norm one function v in Y such that $v(t) = 1$. Then we fix $p = v + x$, $q = x - v$, and $w = (k+1)\, v$. Let $p_1 = p \wedge w$. By construction $p_1 \in B(0, k+1) \cap Y$. If we now take $p_2 = p_1 \vee x$, then $p_2 \in B(0, k+1) \cap B(x, 1) \cap Y$. Now from the fact that $t \in \{s \in K : x(s) \leqslant k\}$ it follows that $p_2(t) = x(t) + 1$. Consequently, we can fix p_2 as the desired p_t.

To define q_t, it is enough to take $q_t = 2x - p_t$. It is straightforward to prove that q_t satisfies the required properties.

Now it is enough to pick up a $t \in K$ for which the functions p_t and q_t exist to conclude that $r(A) = 1$.

To estimate $r_G(A)$ we follow a similar reasoning. First given t in

$$(K_0 \backslash \Omega) \cap \{s \in K : x(s) \geqslant k\},$$

we look for a function u_t such that $u_t(t) = k+1$. By similar reasons to those above this set is also nonempty. Let t be fixed in $(K_0 \backslash \Omega) \cap \{s \in K : x(s) \geqslant k\}$. To define u_t we take u in Y attaining its norm, equal to $k+1$, at t. Then there exists a function p in Y and belonging to $B(x, 1)$ such that $p(t) = x(t) + 1$. Now, it is enough to define $u_t = (p \wedge u) \vee x$. From the fact that there is at least one t satisfying the above conditions we finally obtain $r_G(A) = k+1$. Now (b) implies

$$\lim_{\varepsilon \to 0^+} \operatorname{dist}(E^\varepsilon(A), G) = k. \tag{3.2}$$

We write $K = K^+ \cup K^-$, where $K^- = \{t \in K : x(t) \leqslant k\}$ and $K^+ = \{t \in K : x(t) \geqslant k\}$. Since for all $t \in (K^- \backslash \Omega) \cap K_0$ there exist p_t and q_t as above, and due to the fact that $(K^- \backslash \Omega) \cap K_0$ is dense in K^-, we obtain that $|z(t) - x(t)| \leqslant \varepsilon$ for all $t \in K^-$ and $z \in E^\varepsilon(A)$. Similarly, if $t \in (K^+ \backslash \Omega) \cap K_0$, then there exists u_t in A such that $u_t(t) = k+1$; hence $k - z(t) \leqslant \varepsilon$ for all $t \in K^+$ and $z \in B(0, k) \cap E^\varepsilon(A)$. But from (3.2), it follows that for every $\delta > 0$ there exist $\varepsilon < \delta$ and $z \in Y$ such that $z \in B(0, k+\delta) \cap E^\varepsilon(A)$.

Since $K=K^{-}\cup K^{+}$, we finally get that if $z\in B(0,k+\delta)\cap E^{\varepsilon}(A)$ with $\varepsilon<\delta$ then $\|z-\min\{x,k\}\|\leqslant\delta$. The arbitrariness of δ completes the proof.

(c) $\Rightarrow$ (d) Since X is $\aleph_0$-hyperconvex, X has proper representations. The only thing we must prove is that $\Omega=\varnothing$. Suppose that t_0 is in Ω. We first claim that if $p\in B_Y^{+}=\{x\in Y\colon \|x\|\leqslant 1 \text{ and } x>0\}$, then $p(t_0)=\min_{t\in K}p(t)$. Otherwise we could take w as the minimum of p and the constant function equal to $p(t_0)$, i.e., $w=p\wedge p(t_0)$, which would contradict (c).

Let p in B_Y^{+} be such that $p(t_0)>0$. There s no restriction in assuming that the dimension of X is greater than or equal to two (otherwise the theorem follows trivially), and hence we can find $q\in Y$ not a multiple of p. Then we take a multiple of q, which we will also denote by q, so that there exists $t_1\in K$ with $p(t_1)=q(t_1)$ while $p(t_0)\neq q(t_0)$. Now since $|p-q|$ is positive and belongs to Y, we can define w as a multiple of $|p-q|$ so that w is in B_Y^{+}. But $w(t_0)>\min_{t\in K}w(t)=0$, which contradicts our claim and thus (d) holds.

(d) $\Rightarrow$ (e) X enjoys proper representation by hypothesis. So, let x be a positive element in Y and k a positive number. If $k\geqslant\|x\|$ then $x\wedge k$ is in Y trivially. Let $k\in(0,\|x\|)$. It is enough to prove that for every $\varepsilon>0$ there exists $y\in Y$ such that $\|y-(x\wedge k)\|\leqslant\varepsilon$.

For each $t\in K$ we may fix an open neighborhood in the following way. If $x(t)=0$ then, from the continuity of x, there exists a neighborhood of t such that $x(s)<\varepsilon$ for all s in that neighborhood. For such a t we fix $y_t=x$. If $x(t)>0$ then, by (d), there is a positive element y_t in Y with norm equal to k attaining its norm at t. Thus there is a neighborhood of t such that $0<k-y_t(s)<\varepsilon$ for all s in that neighborhood. Consequently, we have an open covering of K. We consider a finite subcovering and denote by t_i the elements determining the neighborhoods of the subcovering. Let $y=x\wedge(y_{t_1}\vee\cdots\vee y_{t_k})$. Now the proof is complete since either $y(s)=x(s)$, in which case $y(s)$ and $(x\wedge k)(s)$ are both in $(0,\varepsilon)$, or $y(s)=\bigvee_{i=1}^{k}y_{t_i}(s)$, in which case $y(s)\in(k-\varepsilon,k)$ and $x(s)\in(y(s),k)$.

(e) $\Rightarrow$ (a) Let us fix $d>0$. We need to prove that $\tilde{\kappa}_X(d)\geqslant k$ whenever $1<k<d+1$. Since the value of the modulus is invariant under isometries we will calculate it over a proper representation Y of X instead of over X.

Consider x and y as two elements of Y. Without loss of generality we can take y to be the null element of Y. Let $z_x=((k-1)\wedge x^{+})-((k-1)\wedge x^{-})$. From (e) z_x is in Y. But by the construction z_x is such that $\|z_x-0\|\leqslant k-1$, and $B(x,1)\cap B(0,k)\subset B(z_x,1)$. Now, let $\alpha=\frac{k-1}{d}$. Then $0<\alpha<1$ and $\|z_x-0\|\leqslant k-1\leqslant\alpha d$. This proves that $k\leqslant\tilde{\kappa}_X(d)$. ∎

As a consequence of this theorem we obtain our characterization of $C(K)$ and $C_0(K)$ spaces.

THEOREM 3.11. *A Banach space X is a Smith–Ward space if and only if it is isometric to a $C(K)$ or a $C_0(K)$ space.*

Proof. Since $C(K)$ and $C_0(K)$ spaces satisfy the conditions of statement (e) in the last theorem, all these spaces are Smith–Ward spaces. In order to proof the reverse implication let us suppose that X is a Smith–Ward space, and let Y be a proper representation of X. We consider the set $H = \{t \in K : y(t) = 0, \text{ for all } y \in Y\}$. Then it is clear that H is either empty or singleton.

Let us first suppose that H is empty. For each $t \in K$ we fix $x_t \in Y$ such that $x_t(t) = \|x_t\| = 1$. Now, following similar reasoning to that in the proof of (d) implies (e) in the last theorem, we can associate a finite family of open neighborhoods with a finite number of points t_i of K covering K in such a way that there exist functions x_{t_i} satisfying $x_{t_i}(s) \geqslant \frac{1}{2}$ for every s in the open neighborhood associated with t_i. If $z = \bigvee_{1 \leqslant i \leqslant n} x_{t_i}$, then $z(t) \geqslant \frac{1}{2}$ for all $t \in K$. Hence, recalling the last theorem, the constant function equal to $\frac{1}{2}$ is in Y. Now Theorem 2.1 implies that Y is a $C(K)$ space.

To finish the proof let us suppose that $H = \{t_0\}$. Following Theorem 2.2, we denote the set of all triplets determining Y by $\mathfrak{F}$. It will be enough to prove that this is the set of triplets given by Corollary 2.3. Obviously $\mathfrak{F} \supseteq \{(t, t, 1) : t \in K\} \cup \{(t_0, t_0, \alpha) : \alpha \geqslant 0\}$. To obtain the reverse inclusion we fix a triplet (t_1, t_2, α) such that $t_1 \neq t_2$. Then we may take $x \in Y$ such that $x(t_1) > 0$ and $x(t_2) \geqslant 0$ (note that, by the properties of Y, α cannot be equal to 1). Let $z = x \wedge (x(t_1) + x(t_2))/2$. Obviously z is in Y. But if (t_1, t_2, α) is in $\mathfrak{F}$, then $z(t_1) \neq \alpha z(t_2)$. So (t_1, t_2, α) is not in Y and hence the proof is complete. ∎

We conclude this section with the following corollary.

COROLLARY 3.12. *Let X be a Banach space. Then*

$$r_G(A) = r(A) + \operatorname{dist}(E^0(A), G)$$

for every pair of nonempty subsets A and G of X, with A bounded, if and only if X is a Smith–Ward space.

Proof. Since the chain of inequalities

$$r_G(A) \leqslant r(A) + \lim_{\varepsilon \to 0^+} \operatorname{dist}(E^\varepsilon(A), G) \leqslant r(A) + \operatorname{dist}(E^0(A), G)$$

always holds, if X is as in the statement then it is a Smith–Ward space. Conversely, if X is a Smith–Ward space, then, by Theorem 3.11, it is either a $C(K)$ or a $C_0(K)$ space. Now it is enough to recall Corollary 2.11 and the fact, stated in [2], that $E^0(A)$ is a nonempty set for every nonempty bounded subset A of a $C(K)$ or a $C_0(K)$ space to conclude the proof. ∎

4. THE INCOMPLETE CASE

In this section we study the theorems of Section 3 in the context of incomplete normed linear spaces. If Y is a normed linear space then $\bar{Y}$ stands for the completion of Y. The next lemma follows trivially from the definition of being a Smith–Ward space:

LEMMA 4.1. *A normed linear space Y is a Smith–Ward space if and only if its completion is a Smith–Ward space.*

If we recall Theorem 3.10, then we can write:

LEMMA 4.2. *If Y is a normed linear space, then the following assertions are equivalent:*

(a) *Y is a Smith–Ward space.*

(b) *$\tilde{\kappa}_{\bar{Y}}(d) = d+1$ for all positive d.*

The next lemma proves that we cannot replace $\tilde{\kappa}_{\bar{Y}}(d)$ by $\tilde{\kappa}_Y(d)$ in Lemma 4.2.

LEMMA 4.3. *Let c_0 be the space of all real sequences convergent to 0 endowed with the supremum norm. Consider $Y = \{x \in \ell_1 : x_1 = \sum_{i=2}^{+\infty} x_i\}$ as a subspace of c_0 with its induced norm. Then the following assertions hold:*

(a) *Y is dense in c_0. Consequently, since c_0 is a $C_0(K)$ space, Y is a Smith–Ward space.*

(b) *Y is an $\aleph_0$-hyperconvex space.*

(c) *$\tilde{\kappa}_Y(d) = \max\{1, d-1\}$ for all positive d.*

Proof. (a) Given $x \in c_0$ and $\varepsilon > 0$ we look for $y \in Y$ such that $\|x-y\| \leqslant \varepsilon$. Let n_0 be a natural number such that $|x_n| \leqslant \frac{\varepsilon}{2}$ for every $n \geqslant n_0$. It is easy to see that we may fix y_i in the real interval $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ so that there exists a natural number n_1 for which $\sum_{i=2}^{n_0} x_i + \sum_{i=n_0+1}^{n_1} y_i = x_1$. Now defining y by

$$y = \begin{cases} x_i, & \text{if } 1 \leqslant i \leqslant n_0, \\ y_i, & \text{if } n_0 < i \leqslant n_1, \\ 0, & \text{if } i > n_1, \end{cases}$$

the assertion follows.

(b) Let $x^1, ..., x^n$ be n elements of Y and $r_1, ..., r_n$ n positive numbers such that $\|x^i - x^j\| \leqslant r_i + r_j$ for every $i, j \in \{1, ..., n\}$. We must prove that $\bigcap_{i=1}^n B(x^i, r_i) \cap Y \neq \emptyset$. Since c_0 is $\aleph_0$-hyperconvex, we can fix $x \in c_0$

belonging to $\bigcap_{i=1}^{n} B(x^i, r_i)$. From x we construct a sequence y as in (a) but taking on this occasion n_0 such that $|x_j^i| \leqslant r_{i_0}/2$ for every $i \in \{1, ..., n\}$ and $j \geqslant n_0$, where $r_{i_0} = \min\{r_i : 1 \leqslant i \leqslant n\}$. This sequence y will be in the above intersection.

(c) We fix $d \in \mathbb{R}^+$ and $k > \max\{d-1, 1\}$. It is enough to show that it is possible to find two elements x and y of Y such that for every z in Y satisfying $B(y, k) \cap B(x, 1) \subset B(z, 1)$, it must occur that $z = x$.

Let $n \in \mathbb{N}$ be such that $\frac{d}{n} \leqslant k - 1$. Then we take

$$x = \left(d, \frac{d}{n}, (n \text{ times}), \frac{d}{n}, 0, ..., 0, ... \right)$$

and y the constant null sequence. Let us suppose that z is an element of Y satisfying $B(0, k) \cap B(x, 1) \subset B(z, 1)$. From the construction $(-k, k) \supseteq (x_i - 1, x_i + 1)$ for all $i \geqslant 2$. Then for each fixed $i \geqslant 2$ we can find a sequence v of c_0 in $B(0, k) \cap B(x, 1)$ such that $v_i = x_i + 1$. Then we complete v so as to be in $B(0, k) \cap B(x, 1) \cap Y$. Hence z_i must be greater than or equal to x_i for all $i \geqslant 2$. Following similar reasoning we can find a sequence w in $Y \cap B(0, k) \cap B(x, 1)$ such that $w_i = x_i - 1$ for all $i \geqslant 2$. Consequently we get that $x_i = z_i$ for all $i \geqslant 2$. Finally, since z and x are both in Y, the proof is complete. ▮

COROLLARY 4.4. *The Lifschitz modulus $\tilde{\kappa}$ is not invariant under completion.*

In order to able to apply results from the above section to the incomplete case we need to introduce a new modulus which is slightly different from the Lifschitz modulus.

DEFINITION 4.5. Let Y be a normed linear space; then, for every $d \geqslant 0$, the incomplete Lifschitz modulus of Y on d, $\tilde{\kappa}'_Y(d)$, is the supremum of all the real positive numbers k such that there exists a number $\alpha \in (0, 1)$ so that for every x and y in Y, and for every ε and $r \in \mathbb{R}^+$ there exists $z \in Y$ such that $d(z, y) \leqslant \alpha\, dr$ and $B(x, r) \cap B(y, kr) \subseteq B(z, r + \varepsilon)$.

Both moduli are directly related by the following obvious lemma.

LEMMA 4.6. *If Y is a normed linear space, then* $\max\{1, d-1\} \leqslant \tilde{\kappa}_Y(d) \leqslant \tilde{\kappa}'_Y(d) \leqslant d + 1$ *for all positive numbers d.*

The proof of the following theorem is similar to that of [8, Theorem 4.2].

THEOREM 4.7. *Let A and G be two nonempty subsets of a normed linear space Y, with A nonsingleton and bounded. Then*

$$r_G(A) \geqslant r(A)\, \tilde{\kappa}'_Y\left(\frac{d'_r(A)}{r(A)}\right),$$

where $d'_r(A) = \lim_{\varepsilon \to 0^+} \operatorname{dist}(E^\varepsilon(A), G)$.

Proof. Let A and G be as in the statement and denote $r_G(A)$ by k. Since Y is a normed space there is no loss of generality if we assume that $r(A) = 1$.

Let $k < \tilde{\kappa}'_Y(d'_r(A))$. Given $\eta > 0$ we take $0 < \delta \leqslant \eta$ such that

$$\operatorname{dist}(E^{2\delta}(A), G) \geqslant d'_r(A)(1 - \eta). \tag{4.1}$$

Since $r_G(A) = k$, we fix $y \in G$ such that $A \subseteq B(y, (1+\delta)\, k)$ and $x \in E^\delta(A)$. Taking the r and ε in the definition of $\tilde{\kappa}'_Y$ as $(1+\delta)$ and $\frac{\delta}{1+\delta}$, respectively, there exists $\alpha \in (0, 1)$, which does not depend on δ, x, or y, such that there exists $z \in Y$ with the following two properties:

$$B(x, 1+\delta) \cap B(y, (1+\delta)\, k) \subseteq B\left(z, (1+\delta)\left(1 + \frac{\delta}{1+\delta}\right)\right)$$

and

$$\|z - y\| \leqslant \alpha d'_r(A)(1+\delta) \leqslant \alpha d'_r(A)(1+\eta). \tag{4.2}$$

Since $x \in E^\delta(A)$, we obtain

$$A \subseteq B\left(z, (1+\delta)\left(1 + \frac{\delta}{1+\delta}\right)\right) = B(z, 1 + 2\delta),$$

and hence $z \in E^{2\delta}(A)$. If we recall formulae (4.1) and (4.2) we obtain a contradiction to the fact that η is arbitrary and α is fixed and strictly less than one. ∎

The following lemma is an immediate consequence of the definition of $\tilde{\kappa}'$.

LEMMA 4.8. *If Y is a normed linear space then $\tilde{\kappa}'_Y(d) = \tilde{\kappa}'_{\bar{Y}}(d)$ for every $d \geqslant 0$.*

The following remark proves that the different between d'_r of Theorem 4.7 and d_r of Theorem 2.9 is essential.

Remark 4.9. Let Y be the space of Lemma 4.3 and consider $x = (3, 1, 1, 1, 0, 0, ...)$ and y the null element of Y. We take $A = B(0, 1) \cap B(y, 2)$ and

$G=\{0\}$. Then $E^0(A)=\{x\}$. From Lemmas 4.3 and 4.8 we have that $\tilde{\kappa}'_Y$ is maximal and hence

$$r_G(A)=2<r(A)\,\tilde{\kappa}'_Y\left(\frac{\operatorname{dist}(E^0(A),G)}{r(A)}\right)=\tilde{\kappa}'_Y(\operatorname{dist}(E^0(A),G))=4.$$

We finish this work by giving the final property of the incomplete Lifschitz modulus.

THEOREM 4.10. *A normed linear space Y is a Smith–Ward space if and only if $\tilde{\kappa}'_Y(d)=d+1$ for all positive d.*

Proof. The direct implication is an immediate consequence of Lemma 2.10 and Theorem 4.7. In order to prove the reverse we have that, since Y is a Smith–Ward space, $\overline{Y}$ is a Smith–Ward Banach space. Therefore $\tilde{\kappa}_{\overline{Y}}(d)=d+1$ for all positive d. Recalling Lemma 4.6 we obtain that $\tilde{\kappa}'_{\overline{Y}}(d)=d+1$ for all positive d. Now Lemma 4.8 leads to the conclusion. ∎

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